Non-holonomic Constraint Force Postulates

Bharath H M, Indian Institute Of Technology, Kanpur

Abstract

The extended Hamilton's Priciple and other methods proposed to handle non-holonomic constraints are considered. They dont agree with each other. By looking at its consistency with D'Alembert principle for linear non-holonomic constraints, it was claimed in earlier papers that the direct extension of hamilton's principle is incorrect. Non-holonomic Constraints, linear in velocities were considered for this purpose. This paper analyzes these claims, and shows that they are incorrect. And hence it shows that it is theoretically impossible to judge any attempt on non-holonomic constraints to be wrong, as long as they are consistent with the D'Alembertian for holonomic constraints.

I. Introduction

The problem of mechanics with non-holonomic constraints was considered recently in [3, 2] and [5]. [1] and [9], appear to be among the earliest to propose a direct extension of the hamilton's principle to the non-holonomic regime. The same extension was adopted in [12]. However, later papers ([3, 2, 8] to name a few) attempted to show that such an extension is incorrect. [3] appears to be very conclusive regarding this issue. It has ruled out the extension of hamilton's principle usimg a theoritical argument. The present paper shows that such theoritical arguments cannot rule out the extension.

This paper discusses the extension of hamilton's principle along with other attempts to tackle non-holonomic constraints, and carefully examines the argument used to rule out the direct use of hamilton's principle for non-holonomic constraints. Finally, it points out the reason why the argument is incorrect. That slightly alters the conclusions made in [3] regarding the scope of the D'Alembert's principle. These conclusions show that the two ways of tackling non-holonomic problems are uncomparable.

The paper first briefly discusses the classical theory of mechanics, which can be found in any classical text on mechanics[12]. However, the following section-II outlines the theory in a structure convenient for the present purpose. Section III analyzes the various proposals to tackle non-holonomic constraints. Section IV analyzes the argument given against a direct generalization of the Hamilton's

principle, and points out the fallacy in it. Section V summarizes its implications. The mathematical tools required, namely, the variational calculus is put at the end, in the appendix. Einstein's notation is used for summation throughout. Unless explicitly mentioned, repeated indices mean summation throughout.

II. Holonomic Constraints

1. Newton's Laws

For a system of particles Newton's law can be written under a Cartesian system as: (indices not summed)

$$m_i \ddot{x}_i = X_i$$

Where X_i are the total forces. Newton's law says, acceleration is caused by force and only force. Hence, within the framework of newtons laws, nothing other than a force can influence the dynamics of a system. Any constraint which restricts the motion of a system should hence be replaced by an equivalent 'constraint force' which brings in the same effect as the constraint, so that it can be studied using Newton's laws. The equation becomes $m_i \ddot{x}_i = F_i + Q_i$, where F_i are the applied forces andQ_i are the constraint forces. An additional postulate is required to construct the appropriate constraint forces.

2. Constraint Force Postulate (D'Alembert's Principle):

For a constraint of the form $\mathscr{F}(x,t) = 0$, which means, that the system is confined to a time varying hyper surface in the configuration space, the constraint forces Q_i which have the same effect as that of the constraint can be written as,

$$Q_i = \lambda(t) \frac{\partial}{\partial x_i} \mathscr{F} \tag{1}$$

where $\lambda(t)$ is an undetermined function of time. This formulation of the postulate will be shown to be equivalent to the standard D'Alembert's principle stated in terms of virtual work, defined in most standard texts on mechanics([12], for example).

A 'virtual displacement' is a local variation of $x(t)$ consistent with the constraint. i.e., a local variation $\{\delta_i(t)\}$ such that, $\mathscr{F}_{x_i}\delta_i = 0$,assuming that not all $\mathscr{F}_{x_i}$are zero, at a time t. This is a simpler form of the definition given by [16]. It says "virtual displacement is a vector tangential to the constraint manifold" The definition of a virtual displacement is found all texts on mechanics. However, most of them are unclear and ambiguous. Hence we refer to [16]. As it turns out, the major result pointed out by this paper banks on this definition, hence this extra care on the definition. D'Alembert's principle says the virtual work done by the constraint forces vanishes[12]. i.e., $Q_i\delta_i = 0$, at a time

t for all $\{\delta_i(t)\}$ such that $\mathscr{F}_{x_i}\delta_i = 0$ Or, $Q_i = \lambda\mathscr{F}_{x_i}$ where the constant λ might not be the same at a different time t'. This can be in general written as $Q_i = \lambda(t)\mathscr{F}_{x_i}$. Hence equation 1 is equivalent to D'Alembert's principle. The equations of motion can now be written as

$$m_i\ddot{x}_i = F_i + \lambda(t)\mathscr{F}_{x_i} \tag{2}$$

where F_i are the applied forces.

3. Variational Formulation

For a system where the applied forces are conservative, the equation of motion 2 supports a variational formulation. If $\mathscr{L}$ is the Lagrangian, the equations 2 can be written as ([12]):

$$\frac{d}{dt}\mathscr{L}_{\dot{x_i}} - \mathscr{L}_{x_i} = \lambda(t)\mathscr{F}_{x_i}) \tag{3}$$

Since $\mathscr{F}$ is independent of the velocities, $\lambda(t)\mathscr{F}_{x_i} = -\{\frac{d}{dt}\frac{\partial}{\partial\dot{x_i}}(\lambda(t)\mathscr{F}) - \frac{\partial}{\partial x_i}(\lambda(t)\mathscr{F})\}$. Hence the equations can be written as $\frac{d}{dt}\frac{\partial}{\partial\dot{x_i}}(\mathscr{L}+\lambda(t)\mathscr{F}) - \frac{\partial}{\partial x_i}(\mathscr{L}+\lambda(t)\mathscr{F}) = 0$. Which can be compared with equation 21 in the appendix. These equations suggest the Hamilton's principle of stationary action for this category of constraints. The equation 21 along with initial and final condition (i.e, the *boundary value* problem) is an extremization of the action. However, the corresponding *initial value* problem (i.e, the same differential equation with initial position and velocities known, which is the case in all physical situations) is not strictly equivalent to extremization of the action. However, the Hamilton's principle can be formulated with the obtained boundary values as 'action is stationary for fixed boundaries'.

III. Non Holonomic Constraints

The D'Alemberts virtual work postulate stated in the previous section fails to work with nonholonomic constraints. A new constraint force postulate is required to handle non holonomic constraints. Various postulates are available in literature[14]. The major candidates are:

1.) Gauss-Gibbs Principle

The Gauss-Gibbs principle as described in [14, 6], says that the first order variation of the quantity, C, vanishes under allowed Gaussian variations.

$$C = \frac{1}{2}m_i(\ddot{x}_i - \frac{X_i}{m_i})^2 \tag{4}$$

Variations $\{\delta_i(t)\}$ for which δ_i and $\dot{\delta}_i$ vanish at the concerned value of t are called Gaussian variations. Under such variations, the first order variation of C is given by:

$$\delta C = (m_i \ddot{x}_i - X_i)\ddot{\delta}_i \tag{5}$$

This quantity vanishes for all Gaussian variations consistent with constraints. The consistency condition translates as:

$$\mathscr{F}(x+\delta, \dot{x}+\dot{\delta}, t) - \mathscr{F}(x, \dot{x}, t) = \mathscr{F}_{x_i}\delta_i + \mathscr{F}_{\dot{x}_i}\dot{\delta}_i = 0 \tag{6}$$

Which is trivially true at the concerned time t. And hence we turn to its first order variation in time.

$$\delta\mathscr{F}(t+\delta t) - \delta\mathscr{F}(t) = \{\mathscr{F}_{\dot{x}_i}\ddot{\delta}_i + (\dot{\mathscr{F}}_{\dot{x}_i} + \mathscr{F}_{x_i})\dot{\delta}_i + \dot{\mathscr{F}}_{x_i}\delta_i\}\delta t = 0 \tag{7}$$

That puts the condition $\mathscr{F}_{\dot{x}_i}\ddot{\delta}_i = 0$ on the $\ddot{\delta}_i$ (Second order time variation needs to be invoked for holonomic constraints). The principle is now equivalent to newton's laws with a force postulate:

$$Q_i = \lambda(t)\mathscr{F}_{\dot{x}_i}..........(nonholonomic) \tag{8}$$

$$Q_i = \lambda(t)\mathscr{F}_{x_i}............(holonomic) \tag{9}$$

A similar result is obtained in [4].

2.) Jordain Principle

The fundamental equation of the Jourdain principle is([7, 14])

$$(m_i \ddot{x}_i - X_i)\dot{\delta}_i = 0 \tag{10}$$

Where $\delta_i(t)$ are a 'Jourdain' variations which vanish at t, while their derivatives dont. Such variations are called 'virtual velocities'. The constraint force postulate used is same as those in the Gauss's principle(8,9)

3.) Extended Hamilton's principle: Vakonomic Mechanics

Extending Hamilton's principle of stationary action was proposed in [1, 9], later criticized heavily ([2, 3, 8] for example.) Nevertheless, it is equivalent to assuming a constraint force given by

$$Q_i = \lambda(t)\{\mathscr{F}_{x_i} - \frac{d}{dt}\mathscr{F}_{\dot{x}_i}\} + \mathscr{F}_{\dot{x}_i}\frac{d\lambda(t)}{dt} \tag{11}$$

This was termed as 'vakonomic mechanics'(**V**ariational **A**xiomatic **K**ind) by [17] since the basic axiom here is the Hamilton's principle, which is variational in nature.

We now have two possible postulations of the the constraint force: 89 and 11. We have one well accepted formulation 1(D'Alembert's principle) for holonomic constraints. Both 89 and 11 agree with 1 for the special case of holonomic constraints. But, they don't at proper nonholonomic constraints. One curious and simple set of constraints which might be of help in deciding the correctness of 89 and 11 is the linear nonholonomic constraints. This has been used in [2, 3] to rule out 11. However, there seems to be a subtle issue with that and hence we wish to reconsider it.

IV. Linear Non-Holonomic Constraints

Linear nonholonomic constraints are non integrable equations of the form,

$$a_i\dot{x}_i + a_t = 0 \tag{12}$$

Where a_i and a_t are functions independent of $\dot{x}_i$. This category of constraints has been recently considered in [3] to discard 11. [3] has concluded that all nonholonomic constraints are beyond the scope of Hamilton's principle; the general nonholonomic constraints are beyond the scope of D'Alemberts principle, while the linear non holonomic constraints do remain within the scope of D'Alembert's principle. And they agree with what 89 says; and 11 is inconsistent with it, hence it is incorrect. This has been done in other literature as well. [3] examines the reason behind it's conclusions. And it says, the reason why 11 fails at a linear non-holonomic is that the *allowed* variations are *not consistent* with the constraints. By allowed variations, we mean the variations constrained by

$$a_i\delta_i = 0 \tag{13}$$

which is indeed obtained from the constraint equation, but not directly stating that the variations are consistent with the constraint. But, the constraint force given by 13 is $Q_i = \lambda(t)a_i$. Which is same as what is obtained from 89, as mentioned above. However, 89 obtaines the constraint force through a different route, not from 13. Since equation 13 is mysterious, we take a look at how to arrive at 13 from 12.

It is widely mentioned that 13 is obtained from 12 using D'Alembert's principle of virtual work([2, 3, 10, 13, 14, 15]). They write 12 as

$$a_i dx_i + a_t dt = 0 \tag{14}$$

And then replace dx_i by δ_i and dt by 0. But the trouble is, this cannot be done, if we define a D'Alembertian displacement, $\delta_i(t)$ as a variation in $x_i(t)$. dx_i is a differential motion over a time dt, and is given by $dx_i = \dot{x}_i dt$. It vanishes if dt is set to zero. This definition of virtual displacement is used in [16]. A true expression of the constraints on the variation is,

$$a_i\dot{\delta}_i + a_{ix_j}\dot{x}_i\delta_j + a_{tx_j}\delta_j = 0 \tag{15}$$

D'Alembertian displacements don't say anything about the derivatives of the displacements. They are variations in position, by definition. As shown earlier, the same constraint force can be obtained by a different route using Gaussian principle or the Jourdain principle. But the force there is expressed as $Q_i = \lambda(t)\mathscr{F}_{\dot{x}_i}$ for non holonomic constraints, where $\mathscr{F} = a_i\dot{x}_i + a_t$. Moreover, Jourdain displacements are virtual velocities, not virtual displacements. Hence it is impossible to deduce the constraint force $Q_i = \lambda(t)\mathscr{F}_{\dot{x}_i}$ directly from D'Alembert's virtual displacement principle, even for the case of linear non holonomic constraints.

Hence, equation 13, which is the basis of all support to 89 and all criticism on 11 is incorrect. It means, the linear non holonomic too, is outside the scope of D'Alembert's principle, and hence is no way of deciding between 89 and 11. There is no theoretical way to decide between 89 and 11. Neither of them are *intrinsically* incorrect. However, 89 seems to be accepted in most practical examples.

V. Concluding Remarks

To summarize, there are two approaches to mechanics. One based on a constraint force postulate, namely the Jourdain/Gaussian principle(that covers the D'Alembertian also) and the other based on a variational principle, namely, the Hamilton's principle. Mechanics based on Hamilton's principle is called Vakonomic mechanics. More theoretical aspects of the Vakonomic mechanics can be found in[17]. The D'Alembert's principle is the most intuitive among all. It uses the intuitive concept of a constraint force acting instantaneously normal to the constraint surface, and hence forcing the particle to the surface. And hence it is most reliable. Earlier papers have attempted to compare the two approaches by checking their consistency with the D'Alembert's principle. In this paper, we showed that such a comparison is impossible. However, with an application point of view, the constraint approach is found to be more oftenly used.

VI. Appendix: Calculus Of Variation

A detailed account of calculus of variation can be found any standard text on the subject[18, 19, 20] for example). This section contains a brief outline of calculus of variation in a convenient format, including all the results relevant for this paper. This whole section refers to [18, 19].

Calculus is the study of local variations. For a functional $\mathscr{L}$(f,f',t) of a function f(t), and its integral defined as

$$I(f) = \int_{t_1}^{t_2} \mathscr{L}(f, f', t)dt \tag{16}$$

where f' is the time derivative of f. The local variation in $I(f)$ for a local variation of $f_i(t)$ from $f_i(t)$ to $f_i(t)+\delta_i(t)$ is given by

$$I(f+\delta)-I(f)=\int_{t_1}^{t_2}(\mathscr{L}_{f_i}\delta_i+\mathscr{L}_{f_i'}\delta_i')dt+\int_{t_1}^{t_2}(\mathscr{L}_{f_if_j}\delta_i\delta_j+\mathscr{L}_{f_if_j'}\delta_i\delta_j'+\mathscr{L}_{f_i'f_j'}\delta_i'\delta_j')dt \tag{17}$$

up to the second order. Under the additional condition that $f_i(t_1)$and $f_i(t_2)$ are fixed during the variation i.e., $\delta_i(t_1)=\delta_i(t_2)=0$, the first order variation in $I(f)$ can be written as,

$$I(f+\delta)-I(f)=\int_{t_1}^{t_2}\{\mathscr{L}_{f_i}-\frac{\mathrm{d}}{dt}(\mathscr{L}_{f_i'})\}\delta_i dt \tag{18}$$

1.) Unconstrained Extremization

Extremization of I(f) would mean that finding a f(t) where its first order variation vanishes. i.e, $\int_{t_1}^{t_2}\{\mathscr{L}_{f_i}-\frac{\mathrm{d}}{dt}(\mathscr{L}_{f_i'})\}\delta_i dt=0$for all functions δ_i. That is equivalent to the n differential equations $\mathscr{L}_{f_i}-\frac{\mathrm{d}}{dt}(\mathscr{L}_{f_i'})=0$ with the boundary conditions given by $f_i(t_1)$and $f_i(t_2)$

2(a) Constrained Extremization.

We first consider a constraint of the following type:$\int_{t_1}^{t_2}\mathscr{F}(f,f',t)dt=0$ where, $\mathscr{F}(f,f',t)$ is a functional. This problem is now equivalent finding functions $\{f_i(t)\}$ such that,

(i) They satisfy the constraint $\mathscr{F}(f,f',t)$

(ii)First order variation of I(f) vanishes under local variations in$\{f_i(t)\}$ consistent with the constraint $\mathscr{F}(f,f',t)$ and the boundary conditions on f(t)

(iii)$\{f_i(t)\}$satisfy the boundary conditions.

Let $\{\delta_i(t)\}$be a set of n functions and $\{\epsilon_i\}$ be a set of n real numbers. Condition (ii) can be written as:

$$\int_{t_1}^{t_2}\{\mathscr{L}_{f_i}-\frac{\mathrm{d}}{dt}(\mathscr{F}_{f_i'})\}\delta_i\epsilon_i dt=0 \tag{19}$$

for all $\{\epsilon_i\}\in S$ where $S=\{\{\epsilon_i\}:\epsilon_i\int_{t_1}^{t_2}\{\mathscr{F}_{f_i}-\frac{\mathrm{d}}{dt}(\mathscr{F}_{f_i'})\}\delta_i dt=0\}$. For all sets $\{\delta_i(t)\}$. Essentially, S is the set of all variations consistent with the constraints. Such a definition of S will not work if $\mathscr{F}_{f_i}-\frac{\mathrm{d}}{dt}(\mathscr{F}_{f_i'})=0$ whenever $\mathscr{F}=0$ for all i. The second order variation should be invoked in such cases. For a particular set $\{\delta_i\}$, let $\alpha_i=\int_{t_1}^{t_2}\{\mathscr{L}_{f_i}-\frac{\mathrm{d}}{dt}(\mathscr{L}_{f_i'})\}\delta_i dt$ and $\beta_i=\int_{t_1}^{t_2}\{\mathscr{F}_{f_i}-\frac{\mathrm{d}}{dt}(\mathscr{F}_{f_i'})\}\delta_i dt$. S is now the set of n-vectors $\{\epsilon_i\}$ orthogonal to the vector β. Hence the condition

says, $\alpha_i = \lambda\beta_i$ for a real number λ. A simple argument can establish that the number λ is independent of the set $\{\delta_i\}$. Hence, we have the final equations:

$\alpha_i = \lambda\beta_i$ for all sets $\{\delta_i\}$. Or, equivalently,

$$(\mathscr{L} - \lambda\mathscr{F})_{f_i} - \frac{\mathrm{d}}{dt}((\mathscr{L} - \lambda\mathscr{F})_{f_i'}) = 0 \tag{20}$$

λ is the Lagrange multiplier for this case. This is easily generalized to several constraints.

2(b) Constrained Extremization: constraints of second type.

Now we consider constraints of the form $\mathscr{F}(f, f', t) = 0$. These constraints can be equivalently written as

$\int_{t_1}^{t_2} k(t)\mathscr{F}(f, f', t)dt = 0$ for all functions $k(t)$, or equivalently, for $k(t) = k_j(t)$, where $\{k_j(t)\}$ form a basis for functions on $[t_1, t_2]$ Now, the constraint has reduced to the previous category of constraints. The equations may now be written as:

$$(\mathscr{L} - \lambda(t)\mathscr{F})_{f_i} - \frac{\mathrm{d}}{dt}((\mathscr{L} - \lambda(t)\mathscr{F})_{f_i'}) = 0 \tag{21}$$

where $\lambda(t)$ is the Lagrange multiplier for this case.

References

[1] John R. Ray, "Non holonomic Constraints" , Am. J. Phys, 34(406-408) 1966

[2] C. Cronstr öm, T. Raita , "On non-holonomic systems and variational principles", 2008, `http://lanl.arxiv.org/abs/0810.3611`

[3] M. R. Flannery, "Enigma of non holonomic constraints", Am. J. phys, 73(265 - 272), 2005

[4] Eugene Cromer, "A Variational Principle for Non holonomic Constraints", Am. J. Phys 38(1970), Number 7.

[5] C. Cronstr öm, T. Raita, "Existence of Hamiltonians For Non holonomic Systems", `http://lanl.arxiv.org/abs/0711.3997`, 2008

[6] John R. Ray, "Non Holonomic constraints and Gauss's principle of least constraint", Am. J. Phys. 40(1972), Number 1

[7] Li-Sheng Wang, Yih-Hsing Pao, "Jourdain's variational equation and Appell's equation of motion for non holonomic dynamical systems", Am. J. Phys 71(2003), number 1

[8] Errata Am. J. Phys, 34(1966), number 12

[9] G. Goedecke, “Undetermined multiplier treatments of the Lagrange problem”, Am. J. Phys, 34(1966), 571

[10] L. A. pars “Variational Principles in Mechanics” . Quart. jour. Mech and Appl Math, Vol 7, 1954

[11] H. Jefferys, “What is Hamilton’s Principle”,Quart. jour. Mech and Appl Math, Vol 7, 1954

[12] H. Goldstein, C. Poole, J. Safko, *Classical Mechanics*, Addison Wesley

[13] E. T. Whittaker, *Analytical Dynamics Of Particles and Rigid Bodies*, Cambridge university press

[14] L. A. Pars, *A Treatise On Analytical Dynamics*, Heinemann Educational books

[15] D. T. Greenwood, *Classical Dynamics*, Dover Publications

[16] V. I. Arnold, *Mathematical Methods Of Classical Mechanics*, Springer

[17] V. I. Arnold, *Dynamical Systems -III(Encyclopedia of Mathematical Sciences)*, Springer

[18] Richard Courant, Fritz John, *Introduction To Calculus and Analysis, Vol II*, Springer

[19] R. Courant, D. Hilbert, *Methods of Mathematical Physics-I*, Interscience Publishers

[20] R Courant, *Calculus of Variations*, Courant Institute of mathematical sciences